

DIFFERENTIAL SUBORDINATIONS FOR HARMONIC COMPLEX-VALUED FUNCTIONS

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ABSTRACT. Let Ω be any set in the complex plane \mathbb{C} , and let $\psi : \mathbb{C}^3 \times \mathbb{D} \rightarrow \mathbb{C}$. Let p be a complex-valued harmonic function in the unit disc \mathbb{D} of the form $p = p_1 + \overline{p_2}$, where p_1 and p_2 are analytic in \mathbb{D} . In this article we consider the problem of determining properties of the function p , such that p satisfies the differential subordination

$$\psi(p(z), Dp(z), D^2p(z); z) \subset \Omega \Rightarrow p(\mathbb{D}) \subset \Delta.$$

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ be the open disc of the radius r of the complex plane, and let $\mathbb{D} := \mathbb{D}_1$ be the unit disk. Also, we denote by $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}$, and by $\mathcal{Hol}(\mathbb{D})$ the class of holomorphic functions on \mathbb{D} .

A harmonic mapping f of the simply connected region Ω is a complex-valued function of the form

$$f = h + \overline{g}, \tag{1.1}$$

where h and g are analytic in Ω , with $g(z_0) = 0$ for some prescribed point $z_0 \in \Omega$. We call h and g analytic and co-analytic parts of f , respectively. If f is (locally) injective, then f is called (locally) univalent. The Jacobian and the second complex dilatation of f are given by $J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2 = |h'(z)|^2 - |g'(z)|^2$ and $\omega(z) = g'(z)/h'(z)$ ($z \in \Omega$), respectively. A result of Lewy [3] states that f is locally univalent if and only if its Jacobian is never zero, and is sense-preserving if the Jacobian is positive. By $\mathcal{Har}(\mathbb{D})$ we denote the class of complex valued, sense-preserving harmonic mappings in \mathbb{D} . We note that each f of the form (1.1) is uniquely determined by coefficients of the power series expansions [1]

$$h(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n, \quad g(z) = b_0 + \sum_{n=1}^{\infty} b_n z^n \quad (z \in \mathbb{D}), \tag{1.2}$$

where $a_n \in \mathbb{C}$, $n = 0, 1, 2, \dots$ and $b_n \in \mathbb{C}$, $n = 1, 2, 3, \dots$. Following Clunie and Sheil-Small notation [1], we denote by $\mathcal{S}_{\mathcal{H}}$ the subclass of $\mathcal{Har}(\mathbb{D})$, consisting of all sense-preserving

univalent harmonic mappings of \mathbb{D} with the normalization $h(0) = g(0) = h'(0) - 1 = 0$, and its subclass for which $g'(0) = 0$ by \mathcal{S}_H^0 . Several fundamental information about harmonic mappings in the plane can also be found in [2].

For $f \in \mathcal{H}ar(\mathbb{D})$, let the *differential operators* D and \mathfrak{D} be defined as follows

$$Df = z \frac{\partial f}{\partial z} - \bar{z} \frac{\partial f}{\partial \bar{z}} = zh'(z) - \overline{zg'(z)}, \quad (1.3)$$

and

$$\mathfrak{D}f = z \frac{\partial f}{\partial z} + \bar{z} \frac{\partial f}{\partial \bar{z}} = zh'(z) + \overline{zg'(z)}, \quad (1.4)$$

where $\partial f/\partial z$ and $\partial f/\partial \bar{z}$ are the formal derivatives of the function f

$$\frac{\partial f}{\partial z} = \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) \quad \text{and} \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right).$$

Moreover, we define *n-th order differential operator* by recurrence relation

$$\begin{aligned} D^2 f &= D(Df) = zh' - \overline{zg'} + z^2 h'' - \overline{z^2 g''} = Df + z^2 h'' - \overline{z^2 g''}, & D^n f &= D(D^{n-1} f), \\ \mathfrak{D}^2 f &= \mathfrak{D}(\mathfrak{D}f) = zh' + \overline{zg'} + z^2 h'' + \overline{z^2 g''} = \mathfrak{D}f + z^2 h'' + \overline{z^2 g''}, & \mathfrak{D}^n f &= \mathfrak{D}(\mathfrak{D}^{n-1} f). \end{aligned}$$

We note that in the case when f is an analytic function (i.e. $g(z) = 0$), then both D and \mathfrak{D} reduce to the Alexander differential operator zf' .

Now, we present several properties of the differential operators Df and $\mathfrak{D}f$. Some of them follow from the usual rules of differential calculus, therefore the proofs will be omitted.

Proposition 1.1. *Let $\varphi, \psi \in \mathcal{H}ar(\mathbb{D})$ and let the linear differential operators D and \mathfrak{D} be defined by (1.3) and (1.4). Then:*

$$\begin{aligned} (i) \quad & D(\varphi\psi) = \varphi D\psi + \psi D\varphi, & \mathfrak{D}(\varphi\psi) &= \varphi \mathfrak{D}\psi + \psi \mathfrak{D}\varphi, \\ (ii) \quad & D\left(\frac{\varphi}{\psi}\right) = \frac{\psi D\varphi - \varphi D\psi}{\psi^2}, & \mathfrak{D}\left(\frac{\varphi}{\psi}\right) &= \frac{\psi \mathfrak{D}\varphi - \varphi \mathfrak{D}\psi}{\psi^2}, \\ (iii) \quad & D(\varphi \circ \psi) = \frac{\partial \varphi}{\partial \psi} D\psi + \frac{\partial \varphi}{\partial \bar{\psi}} D\bar{\psi}, & \mathfrak{D}(\varphi \circ \psi) &= \frac{\partial \varphi}{\partial \psi} \mathfrak{D}\psi + \frac{\partial \varphi}{\partial \bar{\psi}} \mathfrak{D}\bar{\psi}. \end{aligned}$$

Proposition 1.2. *Let $f \in \mathcal{H}ar(\mathbb{D})$ and let D and \mathfrak{D} be defined by (1.3) and (1.4). Then*

$$\begin{aligned} (a) \quad & D\bar{f} = -\overline{Df}, & \mathfrak{D}\bar{f} &= \overline{\mathfrak{D}f}, \\ (b) \quad & D \operatorname{Re} f = i \operatorname{Im} Df, & \mathfrak{D} \operatorname{Re} f &= \operatorname{Re} \mathfrak{D}f, \\ (c) \quad & D \operatorname{Im} f = -i \operatorname{Re} Df, & \mathfrak{D} \operatorname{Im} f &= \operatorname{Im} \mathfrak{D}f, \\ (d) \quad & D|f| = i|f| \operatorname{Im} \frac{Df}{f}, & \mathfrak{D}|f| &= |f| \operatorname{Re} \frac{\mathfrak{D}f}{f}, \\ (e) \quad & D \arg f = -\operatorname{Re} \frac{Df}{f}, & \mathfrak{D} \arg f &= \operatorname{Im} \frac{\mathfrak{D}f}{f} \quad (f(z) \neq 0), \\ (f) \quad & \operatorname{Re}[Df \overline{\mathfrak{D}f}] &= |z|^2 J_f. \end{aligned}$$

Proposition 1.3. *Let $f \in \mathcal{H}ar(\mathbb{D})$, and let D , \mathfrak{D} be defined by (1.3) and (1.4). Also, let $z = re^{i\theta}$. Then*

$$\frac{\partial f}{\partial \theta} = iDf, \quad r \frac{\partial f}{\partial r} = \mathfrak{D}f, \quad r \frac{\partial}{\partial r} Df = D^2f, \quad (1.5)$$

$$\frac{\partial |f|}{\partial \theta} = -|f| \operatorname{Im} \frac{Df}{f}, \quad \frac{\partial |f|}{\partial r} = \frac{|f|}{r} \operatorname{Re} \frac{\mathfrak{D}f}{f} \quad (f(z) \neq 0), \quad (1.6)$$

$$\frac{\partial}{\partial \theta} \arg f = \operatorname{Re} \frac{Df}{f} = \operatorname{Re} \frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} \quad (f(z) \neq 0), \quad (1.7)$$

$$\frac{\partial}{\partial \theta} \arg f = \frac{1}{r} \operatorname{Im} \frac{\mathfrak{D}f}{f} = \frac{1}{r} \operatorname{Im} \frac{zh'(z) + \overline{zg'(z)}}{h(z) + \overline{g(z)}} \quad (f(z) \neq 0). \quad (1.8)$$

Remark 1.1. *If $G \in \mathcal{H}ar(\mathbb{D})$, then $DG(z\bar{z}) = 0$ and $\mathfrak{D}G(\arg z) = 0$. Therefore the constant functions for the operators D and \mathfrak{D} are the functions of the form $G(|z|^2)$ and $G(\arg z)$, respectively.*

Remark 1.2. *Let f be a linear transformation of the form $f(z) = \alpha z + \beta \bar{z}$, $\alpha, \beta \in \mathbb{C}$. Then $\mathfrak{D}f(z) = \alpha z + \beta \bar{z} = f(z)$.*

In the present paper we will concentrate on the theory of differential subordination for harmonic functions, similar as known from the theory of analytic functions. A crucial result of this theory is Jack's Lemma, extended later by Miller and Mocanu [5], below.

Lemma 1.1. [5, 6] *Let $z_0 = r_0 e^{i\theta_0}$ with $0 < r_0 < 1$, and let f be an analytic function in \mathbb{D} , continuous on \mathbb{T} with $f(z) \neq 0$. If $|f(z_0)| = \max\{|f(z)| : z \in \overline{\mathbb{D}_{r_0}}\}$, then there exists a number $m \in \mathbb{R}$, $m \geq n \geq 1$ such that $\frac{Df(z_0)}{f(z_0)} = \frac{z_0 f'(z_0)}{f(z_0)} = m$, and $\operatorname{Re} \frac{D^2 f(z_0)}{Df(z_0)} = \operatorname{Re} \left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right) \geq m$.*

2. FUNDAMENTAL LEMMAS

For two analytic functions f and F defined in the open unit disk \mathbb{D} with $f(0) = F(0) = 0$, f is subordinate to F , written $f \prec F$ (or $f(z) \prec F(z)$, $z \in \mathbb{D}$), if there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ in \mathbb{D} , such that $f(z) = F(w(z))$. It is known (see, for example [7, p. 36]) that, if F is univalent in \mathbb{D} , then $f \prec F$ if and only if $f(0) = F(0)$ and $f(\mathbb{D}) \subset F(\mathbb{D})$.

A natural extension of a subordination to complex-valued harmonic functions f and F in \mathbb{D} with $f(0) = F(0)$ is to say f is subordinate to F if $f(z) = F(w(z))$ where w is analytic in \mathbb{D} , $w(0) = 0$ and $|w(z)| < 1$ in \mathbb{D} . However, in the case of complex-valued harmonic functions, an analytic function w must preserve harmonicity and, even $f(\mathbb{D}) \subset F(\mathbb{D})$ and F is univalent, such function w may not exist, as in the case of analytic functions. In their famous paper [1] Clunie and Sheil-Small introduced a notion of a subordination for harmonic functions as follows. We say that f is subordinate to

F if $f, F \in \mathcal{Har}(\mathbb{D})$, F is univalent, and if there is a function w , analytic and univalent in \mathbb{D} with $w(0) = 0, w'(0) > 0$, such that $F(z) = f(w(z))$. In 2000 Schaubroeck [8] introduced a notion of a weak subordination: if f and F are harmonic functions in \mathbb{D} with $f(0) = F(0) = 0$, f is called *weakly subordinate* to F if $f(\mathbb{D}) \subset F(\mathbb{D})$. See [8] for the results relating to this definition.

For our purposes we introduce a notion of a strong subordination.

Definition 2.1. Let $f, F \in \mathcal{Har}(\mathbb{D})$ with $f(0) = F(0)$. Also, let F be univalent in \mathbb{D} , and $f(\mathbb{D})$ be a simply-connected domain. We say that f is *strongly subordinate* to F , if there exists a function w , analytic and univalent in \mathbb{D} with $w(0) = 0, w'(0) > 0$, $|w(z)| < 1$ in \mathbb{D} , and such that $f(z) = F(w(z))$.

We note that such w preserve harmonicity; if F has a dilatation ω , then a dilatation of f is $\omega \circ w$ that satisfies $|\omega \circ w| < 1$. Since $J_f = |w'|^2 J_F$, then f is sense preserving (reversing), if F is sense preserving (reversing). Also, we have $F(0) = f(0) \in \Omega$ when $f(\mathbb{D}) = \Omega \subset F(\mathbb{D})$, and Ω is a simply connected domain. Then, there is a unique conformal and univalent mapping w of \mathbb{D} onto $F^{-1}(\Omega)$, satisfying $w(0) = w(F^{-1}(0)) = 0$ and $|w(z)| < 1$ in \mathbb{D} . For such w it holds $f(z) = F(w(z))$. Therefore, we proved a similar condition as in the analytic case, below.

Theorem 2.1. Let $f, F \in \mathcal{Har}(\mathbb{D})$ and let F be univalent in \mathbb{D} and $f(\mathbb{D})$ be a simply-connected domain. Then f is strongly subordinate F if and only if $f(0) = F(0)$ and $f(\mathbb{D}) \subset F(\mathbb{D})$.

Definition 2.2. By Q we denote the set of functions $q(z) = q_1 + \overline{q_2}$, harmonic complex-valued and univalent on $\overline{\mathbb{D}} \setminus E(q)$, where

$$E(q) = \left\{ \zeta \in \mathbb{T} : \lim_{z \rightarrow \zeta} q(z) = \infty \right\}.$$

Moreover, we assume that $Dq(\zeta) \neq 0$, for $\zeta \in \mathbb{T} \setminus E(q)$. The set $E(q)$ is called an *exception set*.

We note that the functions $q(z) = \overline{z}$, $q(z) = \frac{1+\overline{z}}{1-z}$ are in Q , therefore Q is a non-empty set.

Lemma 2.1. Let p, q harmonic in \mathbb{D} with $p(0) = q(0)$, $p(z) \neq 1$, and let $p(\mathbb{D})$ be a simply connected domain and let $q \in Q$ be univalent in \mathbb{D} . If there exist points $z_0 = r_0 e^{i\theta_0} \in \mathbb{D}$, and $\zeta_0 \in \mathbb{T} \setminus E(q)$, such that $p(z_0) = q(\zeta_0)$ and $p(\overline{\mathbb{D}}_{r_0}) \subset q(\overline{\mathbb{D}}) \setminus E(q)$, then there exists a real number $m \geq 1$, such that

$$Dp(z_0) = m Dq(\zeta_0) \quad \text{and} \quad \operatorname{Re} \frac{D^2 p(z_0)}{Dp(z_0)} \geq m \operatorname{Re} \frac{D^2 q(\zeta_0)}{Dq(\zeta_0)}.$$

Proof. Let $z_0 = r_0 e^{i\theta_0} \in \mathbb{D}$. Since $p(\mathbb{D})$ is a simply connected domain the set $p(\mathbb{D}_{r_0})$ is bounded and $p(\overline{\mathbb{D}}_{r_0}) \subset q(\overline{\mathbb{D}}) \setminus E(q)$. Let $q^{-1}(p(\overline{\mathbb{D}}_{r_0})) = U \subset \mathbb{D}$. Then, there exists

an analytic and univalent function w such, that $w(\mathbb{D}) = U$, $w(0) = 0$ and $w(z_0) = q^{-1}(p(z_0)) = \zeta_0$. Thus $|w(z_0)| = |\zeta_0| = 1$, and by the maximum principle we then have that

$$|w(z)| \leq 1 \quad (z \in \overline{\mathbb{D}}_{r_0}).$$

Using Lemma 1.1 we have

$$\frac{Dw(z_0)}{w(z_0)} = \frac{z_0 w'(z_0)}{w(z_0)} = m, \quad \text{and} \quad \operatorname{Re} \frac{D^2 w(z_0)}{Dw(z_0)} = \operatorname{Re} \left(1 + \frac{z_0 w''(z_0)}{w'(z_0)} \right) \geq m. \quad (2.1)$$

Since $q(w(z)) = p(z)$ ($z \in \overline{\mathbb{D}}_{r_0}$), so that for $z = re^{i\theta}$ ($0 \leq r \leq 1, \theta \in [0, 2\pi]$), we obtain

$$p(re^{i\theta}) = q(w(re^{i\theta})). \quad (2.2)$$

Differentiating (2.2) with respect to θ , and using (1.5), we have

$$iDp(z) = i \frac{zw'(z)}{w(z)} Dq(w(z))$$

or, equivalently

$$Dp(z) = \frac{zw'(z)}{w(z)} Dq(w(z)). \quad (2.3)$$

For the case, when $z = z_0$, it holds $w(z_0) = \zeta_0$ and $zw'/w = m$ by (2.1), hence the above becomes

$$Dp(z_0) = mDq(\zeta_0), \quad (2.4)$$

and the first relation follows.

Now, we differentiate (2.3), with respect to r . Then, by (1.5), we have

$$r \frac{\partial}{\partial r} Dp(z) = D^2 p(z) = r \frac{\partial}{\partial r} \left[\frac{zw'(z)}{w(z)} Dq(w(z)) \right], \quad (2.5)$$

so that we obtain

$$D^2 p(z) = \frac{zw'}{w} \left\{ \left(1 + \frac{zw''}{w'} \right) - \frac{zw'}{w} \right\} Dq(w) + \left(\frac{zw'}{w} \right)^2 D^2 q(w).$$

For $z = z_0$ (then $w(z_0) = \zeta_0$, and $z_0 w'(z_0)/w(z_0) = m$ resp.) the above becomes

$$D^2 p(z_0) = m \left\{ \left(1 + \frac{z_0 w''(z_0)}{w'(z_0)} \right) - m \right\} Dq(\zeta_0) + m^2 D^2 q(\zeta_0).$$

Therefore, dividing both sides of the above by $Dp(z_0)$, and applying (2.4), we obtain

$$\frac{D^2 p(z_0)}{Dp(z_0)} = \left(1 + \frac{z_0 w''}{w'} \right) - m + m \frac{D^2 q(\zeta_0)}{Dq(\zeta_0)}.$$

Taking the real part of both sides, and observing that $\operatorname{Re} \left(1 + \frac{z_0 w''(z_0)}{w'(z_0)} \right) \geq m$, we get

$$\operatorname{Re} \frac{D^2 p(z_0)}{Dp(z_0)} \geq m \frac{D^2 q(\zeta_0)}{Dq(\zeta_0)}, \quad (2.6)$$

and the second assertion follows. \square

Corollary 2.1. *If $p, q \in \mathcal{H}ol(\mathbb{D})$, with $p(0) = q(0) = 1$, $p(z) \not\equiv 1$, and let $q \in Q$ be univalent in \mathbb{D} . If there exist points $z_0 \in \mathbb{D}$ and $\zeta_0 \in \mathbb{T} \setminus E(q)$ such that $p(z_0) = q(\zeta_0)$ and $p(\overline{\mathbb{D}}_{r_0}) \subset q(\overline{\mathbb{D}}) \setminus E(q)$ ($r_0 = |z_0|$), then there exists a real number $m \geq 0$, such that*

$$z_0 p'(z_0) = m \zeta_0 q'(\zeta_0), \quad \text{and} \quad \operatorname{Re} \left(\frac{z_0 p''(z_0)}{p'(z_0)} + 1 \right) \geq m \operatorname{Re} \left(\frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1 \right).$$

Lemma 2.2. *Let $p, q \in \mathcal{H}ar(\mathbb{D})$, $p(\mathbb{D})$ be simply connected and let q be univalent in \mathbb{D} . Also, let $q \in Q$, with $p(0) = q(0) = 1$, $p(z) \not\equiv 1$. If p is not strongly subordinate to q , then there exist points $z_0 = r_0 e^{i\theta_0}$ and $\zeta_0 \in \mathbb{T} \setminus E(q)$ and a number $m \geq 1$ such that $p(\overline{\mathbb{D}}_{r_0}) \subset q(\overline{\mathbb{D}})$, $p(z_0) = q(\zeta_0)$, and*

$$(i) \quad Dp(z_0) = m Dq(\zeta_0);$$

$$(ii) \quad \operatorname{Re} \frac{D^2 p(z_0)}{Dp(z_0)} \geq m \operatorname{Re} \frac{D^2 q(\zeta_0)}{Dq(\zeta_0)}.$$

Proof. Since $p(0) = q(0)$ and $p(z) \not\equiv q(z)$, then $p(\mathbb{D}) \not\subset q(\mathbb{D})$ and $p(\mathbb{D}) \cap q(\mathbb{D}) \neq \emptyset$. Let $r_0 = \sup\{r : p(\overline{\mathbb{D}}_r) \subset q(\overline{\mathbb{D}})\}$. Then we have $p(\overline{\mathbb{D}}_{r_0}) \subset q(\overline{\mathbb{D}})$. Since $p(\overline{\mathbb{D}}_{r_0}) \not\subset q(\mathbb{D})$, and $p(\overline{\mathbb{D}}_{r_0}) \subset q(\overline{\mathbb{D}})$, there exists $z_0 \in \overline{\mathbb{D}}_{r_0}$ such that $p(z_0) \in \partial q(\mathbb{D})$. This implies that there exists $\zeta_0 \in \mathbb{T} \setminus E(q)$ such that $p(z_0) = q(\zeta_0)$. The remaining two conclusions follow by applying Lemma 2.1. \square

Definition 2.3. *Let $\Omega \subset \mathbb{C}$, let $q \in \mathcal{H}ar(\mathbb{D})$. By $\Psi[\Omega, q]$ we denote the class of functions $\psi : \mathbb{C}^3 \times \mathbb{D} \rightarrow \mathbb{C}$ which satisfy the condition:*

$$\psi(r, s, t; z) \notin \Omega, \quad \text{when } r = q(\zeta), \quad s = m Dq(\zeta),$$

$$\operatorname{Re} \left(\frac{t}{s} + 1 \right) \geq m \operatorname{Re} \frac{D^2 q(\zeta)}{Dq(\zeta)},$$

where $z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D} \setminus E(q)$ and $m \geq 1$.

The set $\Psi[\Omega, q]$ is called the class of admissible harmonic functions and conditions are called the harmonic admissibility conditions.

Theorem 2.2. *Let $\psi \in \Psi[\Omega, q]$, $q \in \mathcal{H}ar(\mathbb{D})$ with $q(0) = 1$, and let q be univalent in \mathbb{D} . If $p \in \mathcal{H}ar(\mathbb{D})$ with $p(0) = 1$, $p(\mathbb{D})$ is simply connected and satisfies the condition*

$$\psi(p(z), Dp(z), D^2 p(z); z) \in \Omega \quad (z \in \mathbb{D}), \quad (2.7)$$

then $p \prec q$ in \mathbb{D} .

Proof. We assume that $p \not\prec q$ in \mathbb{D} . From Lemma 2.2 we have that there exist points $z_0 \in \mathbb{D}$ and $\zeta_0 \in \partial\mathbb{D} \setminus E(q)$ and a number $m \geq 1$ such that $p(z_0) = q(\zeta_0)$, and

$$Dp(z_0) = m Dq(\zeta_0), \quad \operatorname{Re} \frac{D^2 p(z_0)}{Dp(z_0)} \geq m \operatorname{Re} \frac{D^2 q(\zeta_0)}{Dq(\zeta_0)}.$$

Using above with $z = z_0$, $r = p(z_0)$, $s = Dp(z_0)$, $t = D^2p(z_0)$ in Definition 2.3 we obtain

$$\psi(p(z_0), Dp(z_0), D^2p(z_0); z_0) \notin \Omega. \quad (2.8)$$

Since (2.8) contradicts (2.7) we have that the assumption made is false, hence $p \prec q$ in \mathbb{D} . \square

Remark 2.1. *In the hypothesis of Theorem 2.2 we have assumed that the behavior of q is known on the boundary of \mathbb{D} . If we don't know the behavior of q on the boundary of \mathbb{D} then we may also prove that $p \prec q$ using the following limit procedure.*

Theorem 2.3. *Let $\Omega \subset \mathbb{C}$, let $q \in \mathcal{H}ar(\mathbb{D})$ with $q(0) = 1$, and let q be univalent in \mathbb{D} , $\psi \in \Psi[\Omega, q_\rho]$ for some $\rho \in (0, 1)$, where $q_\rho(z) = q(\rho z)$. If $p \in \mathcal{H}ar(\mathbb{D})$ and $p(\mathbb{D})$ is a simply connected, with $p(0) = 1$, then*

$$\{\psi(p(z), Dp(z), D^2p(z); z) : z \in \mathbb{D}\} \subset \Omega$$

implies $p \prec q$ in \mathbb{D} .

Proof. Since $q_\rho(z) = q(\rho z)$ we have that the function q_ρ is injective on $\overline{\mathbb{D}}$, hence $E(q_\rho) = \emptyset$ and $q_\rho \in Q$. The function $\psi \in \Psi[\Omega, q_\rho]$ is an admissible function and

$$\{\psi(p(z), Dp(z), D^2p(z); z) : z \in \mathbb{D}\} \subset \Omega,$$

so that, in view of Theorem 2.2 we have that

$$p \prec q \quad (z \in \mathbb{D}). \quad (2.9)$$

On the other hand, $q_\rho(z) = q(\rho z)$ implies that

$$q_\rho(z) \prec q(z) \quad (z \in \mathbb{D}). \quad (2.10)$$

From (2.9) and (2.10) we obtain $p(z) \prec q_\rho(z) \prec q(z)$ which gives $p \prec q$ in \mathbb{D} . \square

Remark 2.2. *Let $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$ be a simply connected domain and η be a harmonic, univalent function in \mathbb{D} . If we assume that $\eta(\mathbb{D}) = \Omega$, then by letting $\Psi[\eta, q] \equiv \Psi[\eta(\mathbb{D}), q]$ and in view of Theorem 2.2, we obtain the following result.*

Theorem 2.4. *Let $\eta \in \mathcal{H}ar(\mathbb{D})$, be univalent in \mathbb{D} , $\eta(0) = 1$ with $\eta(\mathbb{D}) = \Omega$. Let $q \in \mathcal{H}ar(\mathbb{D})$ be univalent in \mathbb{D} , $q(0) = 1$, $q(\mathbb{D}) = \Delta$, and $\psi(p(z), Dp(z), D^2p(z); z)$ be a harmonic function such that $\psi(1, 0, 0; 0) = \eta(0) = 1$, then*

$$\psi(p(z), Dp(z), D^2p(z); z) \prec \eta(z) \quad (z \in \mathbb{D})$$

implies

$$p(z) \prec q(z) \quad (z \in \mathbb{D}).$$

This result can be extended for the case when the behavior of q on $\partial\mathbb{D}$ is not known.

Theorem 2.5. Let $\eta \in \mathcal{H}ar(\mathbb{D})$, univalent in \mathbb{D} , with $\eta(\mathbb{D}) = \Omega$, and let $q \in \mathcal{H}ar(\mathbb{D})$ with $q(0) = 1$, $q(\mathbb{D}) = \Delta$. We let $\eta_\rho(z) = \eta(\rho z)$, $q_\rho(z) = q(\rho z)$. Let $\psi : \mathbb{C}^3 \times \mathbb{D} \rightarrow \mathbb{D}$ be harmonic in \mathbb{D} with $\psi(1, 0, 0; 0) = 1$ and satisfy one of the following conditions:

- (i) $\psi \in \Psi[\eta, q_\rho]$ for some $\rho \in (0, 1)$;
- (ii) there exists a certain $\rho_0 \in (0, 1)$ such that $\psi \in \Psi[\eta_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.

If a function p is harmonic in \mathbb{D} , and $\psi(p(z), Dp(z), D^2p(z); z)$ is harmonic and univalent in \mathbb{D} , then

$$\psi(p(z), Dp(z), D^2p(z); z) \prec \eta(z)$$

implies $p \prec q$ in \mathbb{D} .

Proof. From Theorem 2.2 we have $p(z) \prec q_\rho(z)$. On the other hand $q_\rho(z) \prec q(z)$ for $\rho \in (0, 1)$. From $p(z) \prec q_\rho(z) \prec q(z)$ we have that $p(z) \prec q(z)$, $z \in \mathbb{D}$. If we let $p_\rho(z) = p(\rho z)$, then

$$\psi(p_\rho(z), Dp_\rho(z), D^2p_\rho(z); \rho z) = \psi(p(\rho z), Dp(\rho z), D^2p(\rho z); \rho(z)) \in \eta_\rho(\mathbb{D}).$$

Applying Theorem 2.2 we obtain $p_\rho(z) \prec q_\rho(z)$ for all $\rho \in (\rho_0, 1)$. Next, letting $\rho \rightarrow 1$ we obtain $p \prec q$ in \mathbb{D} . \square

Let $M_1, M_2 > 0$ be such that $M_1 > M_2$. Consider the function $q(z) = 1 + M_1z + M_2\bar{z}$. We note that the function $q(z) = 1 + M_1z + M_2\bar{z}$ maps an unit disk onto an ellipse with the axis $M_1 + M_2$ and $M_1 - M_2$ (see Fig. 1.), q is univalent and sense preserving ($J_q = M_1^2 - M_2^2 > 0$). Moreover $E(q) = \emptyset$.

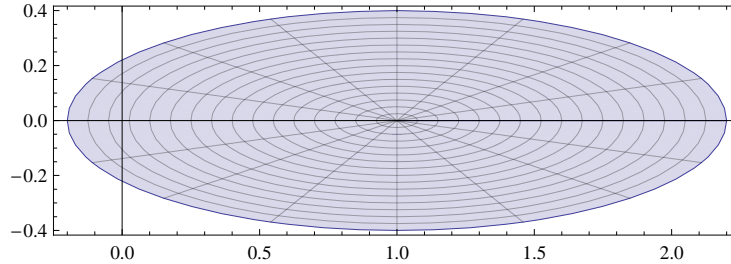


Fig. 1. Image of the unit disk under $q(z)$, with $M_1 = 0.8, M_2 = 0.4$.

Example 2.1. Let $M_1 > M_2 > 0$. If the function p is harmonic in \mathbb{D} , $p(0) = 1$, then

$$p(z) + Dp(z) \prec 1 + M_1z + M_2\bar{z} \Rightarrow p(z) \prec 1 + M_1z + M_2\bar{z}.$$

Proof. Let $\psi(r, s, t; z) = r + s$, and $\eta(z) = q(z) = 1 + M_1z + M_2\bar{z}$. Our proof starts with the observation that $\psi \in \Psi[\eta, q]$. Indeed, for $r = q(\zeta) = 1 + M_1\zeta + M_2\bar{\zeta}$, $s =$

$mDq(\zeta) = m(M_1\zeta - M_2\bar{\zeta})$, and $\zeta \in \partial\mathbb{D}$, we obtain

$$\begin{aligned} |\psi(r, s, t; z) - 1| &= |(m+1)M_1\zeta + (1-m)M_2\bar{\zeta}| \\ &\geq (m+1)M_1 - (m-1)M_2 \\ &= m(M_1 - M_2) + M_1 + M_2 \\ &= M_1 + M_2. \end{aligned}$$

Hence, $\psi(r, s, t; z) \notin \eta(\mathbb{D})$, therefore $\psi \in \Psi[\eta, q]$, and by the Theorem 5.1 the required subordination follows. \square

In the same manner we can see that

Example 2.2. Let $M_1 > M_2 > 0$. If the function p is harmonic in \mathbb{D} , $p(0) = 1$, then

$$p(z) + Dp(z) \prec 1 + 2M_1z \Rightarrow p(z) \prec 1 + M_1z + M_2\bar{z}.$$

We note that setting $p(z) = q(z) = 1 + M_1z + M_2\bar{z}$ we obtain the equality on the left hand side of the implication, that is q is a solution of the above differential subordination.

A slight change in the proof shows that:

Example 2.3. Let $0 < M_2 < \frac{\sqrt{33}-5}{4}M_1$. If the function p is harmonic in \mathbb{D} , $p(0) = 1$, then

$$|p(z) + Dp(z) + D^2p(z) - 1| < M_2 \Rightarrow p(z) \prec 1 + M_1z + M_2\bar{z}.$$

Proof. We proceed as in the previous examples, with $\psi(r, s, t; z) = r + s + t$, and $q(z) = 1 + M_1z + M_2\bar{z}$, $\Omega = \{w : |w - 1| < M_2\}$. Then we have

$$\begin{aligned} |\psi(r, s, t; z) - 1| &= \left| Dp(z) \left(1 + \frac{D^2p(z)}{Dp(z)} \right) + p(z) - 1 \right| \\ &\geq m|Dq(\zeta)| \left| 1 + \frac{D^2p(z)}{Dp(z)} \right| - |q(\zeta) - 1| \\ &\geq m|M_1\zeta - M_2\bar{\zeta}| \operatorname{Re} \left(1 + \frac{D^2p(z)}{Dp(z)} \right) - |M_1\zeta - M_2\bar{\zeta}| \\ &\geq m(M_1 - M_2) \left(1 + m \operatorname{Re} \frac{D^2q(\zeta)}{Dq(\zeta)} \right) - (M_1 + M_2) \\ &\geq m(M_1 - M_2) \left(1 + m \frac{M_1 - M_2}{M_1 + M_2} \right) - (M_1 + M_2) \\ &\geq (M_1 - M_2) \left(1 + \frac{M_1 - M_2}{M_1 + M_2} \right) - (M_1 + M_2) \\ &> M_2, \end{aligned}$$

for $0 < M_2 < \frac{\sqrt{33}-5}{4}M_1$. Therefore, $\psi(r, s, t; z) \notin \Omega$, and by Theorem 5.1 we deduce the assertion. \square

Set now $q(z) = \frac{1+z}{1-z} + \frac{\bar{z}}{1-\bar{z}}$. Then $q(\mathbb{D})$ is a half-plane $\operatorname{Re} w > -\frac{1}{2}$ (see Fig. 2.), $J_q > 0$, $E(q) = \{1\}$.

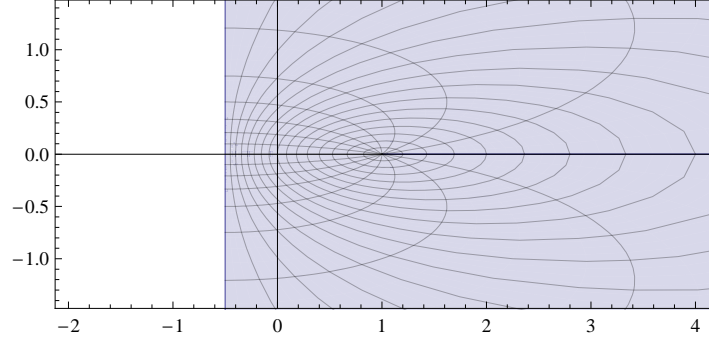


Fig. 2. Image of the unit disk under $q(z)$.

Example 2.4. If the function p is harmonic in \mathbb{D} , $p(0) = 1$, then

$$p(z) + Dp(z) \prec \frac{1+z}{1-z} + \frac{\bar{z}}{1-\bar{z}} \Rightarrow p(z) \prec \frac{1+z}{1-z} + \frac{\bar{z}}{1-\bar{z}}.$$

Proof. We now proceed analogously to the proof of the example 2.1. Set $\psi(r, s, t; z) = r + s$, and

$$\eta(z) = q(z) = \frac{1+z}{1-z} + \frac{\bar{z}}{1-\bar{z}}.$$

We have

$$Dq(z) = D\eta(z) = \frac{2z}{(1-z)^2} - \frac{\bar{z}}{(1-\bar{z})^2},$$

and thus

$$\begin{aligned} q(z) + Dq(z) &= \frac{1+z}{1-z} + \frac{\bar{z}}{1-\bar{z}} + \frac{2z}{(1-z)^2} - \frac{\bar{z}}{(1-\bar{z})^2} \\ &= \frac{1+z}{1-z} + \frac{\bar{z}}{1-\bar{z}} + \frac{z}{(1-z)^2} + 2i\operatorname{Im} \frac{z}{(1-z)^2}. \end{aligned}$$

For $r = q(\zeta)$, $s = mDq(\zeta)$ we obtain

$$\begin{aligned} \operatorname{Re} \psi(r, s, t; z) &= \operatorname{Re} \left(\frac{1+\zeta}{1-\zeta} + \frac{\bar{\zeta}}{1-\bar{\zeta}} + \frac{2m\zeta}{(1-\zeta)^2} - \frac{\overline{m\zeta}}{(1-\bar{\zeta})^2} \right) \\ &= \operatorname{Re} \left(\frac{1+\zeta}{1-\zeta} + \frac{\bar{\zeta}}{1-\bar{\zeta}} + \frac{m\zeta}{(1-\zeta)^2} \right) \\ &< -\frac{1}{2} - m\frac{1}{4} < -\frac{1}{2}. \end{aligned}$$

Hence, $\psi(r, s, t; z) \notin \eta(\mathbb{D})$ therefore $\psi \in \Psi[\eta, q]$. Applying the Theorem 5.1 we have the desired conclusion. \square

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests regarding the publication of this paper.

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